ELSEVIER
Discrete Mathematics 132 (1994) 65–73
DISCRETE MATHEMATICS

Steiner distance stable graphs

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Received 11 June 1991; revised 27 November 1991

Abstract

Let G be a connected graph and S a nonempty set of vertices of G. Then the Steiner distance $d_G(S)$ of S is the smallest number of edges in a connected subgraph of G that contains S. Let k, l, s and m be nonnegative integers with $m \geqslant s \geqslant 2$ and k and l not both 0. Then a connected graph G is said to be k-vertex l-edge (s,m)-Steiner distance stable, if for every set S of s vertices of G with $d_G(S)=m$, and every set A consisting of at most k vertices of $G-S$ and at most l edges of G, $d_{G-A}(S)=d_G(S)$. It is shown that if G is k-vertex l-edges (s,m)-Steiner distance stable, then G is k-vertex l-edge $(s,m+1)$-Steiner distance stable. Further, a k-vertex l-edge (s,m)-Steiner distance stable graph is shown to be a k-vertex l-edge $(s-1,m)$-Steiner distance stable graph for $s \geqslant 3$. It is then shown that the converse of neither of these two results holds.

If G is a connected graph and S an independent set of s vertices of G such that $d_G(S)=m$, then S is called an $I(s,m)$-set. A connected graph is k-vertex l-edge $I(s,m)$-Steiner distance stable if for every $I(s,m)$-set S and every set A of at most k vertices of $G-S$ and l-edges of G, $d_{G-A}(S)=m$. It is shown that a k-vertex l-edge $I(3,m)$-Steiner distance stable graph, $m \geqslant 4$, is k-vertex l-edge $I(3,m+1)$-Steiner distance stable.

1. Introduction

In [1] a connected graph is defined to be *vertex* (*edge*) *distance stable* if the distance between nonadjacent vertices is unchanged after the deletion of a vertex (edge) of G. A more general definition of an equivalent concept was introduced and studied in [4]. It was shown in [4] that a graph is vertex distance stable if and only if it is edge distance stable. We will thus refer to vertex or edge distance stable graphs as distance stable graphs. From a more general result established in [4], it can be deduced that a graph is distance stable if and only if for every pair u, v of nonadjacent vertices, $|N(u) \cap N(v)|=0$ or $|N(u) \cap N(v)| \geqslant 2$. Thus a graph G is distance stable if and only if

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SSDI 0012-365X(93)E000-O

distances between pairs of vertices (in G) at distance 2 apart remain unchanged after the deletion of a vertex or an edge.

Further generalizations of distance stable graphs are studied in [6]. In particular for nonnegative integers k and l not both zero and $D \subseteq N-\{1\}$ a connected graph G is defined to be (k,l,D)-*stable* if for every pair u,v of vertices of G that are at distance $d_G(u,v) \in D$ apart and every set A consisting of at most k vertices (of $G-\{u,v\}$) and at most l edges (of G), the distance between u and v in $G-A$ equals $d_G(u,v)$. For a positive integer m, let $N_{\geqslant m}=\{x \in N \mid x \geqslant m\}$. In [6] it is established that a graph is $(k,l,\{m\})$-stable if and only if it is $(k,l,N_{\geqslant m})$-stable. It is further shown that for a positive integer x a graph is $(k+x,l,\{2\})$-stable if and only if it is $(k,l+x,\{2\})$-stable, but that $(k,l+x,\{m\})$-stable graphs need not be $(k+x,l,\{m\})$-stable for $m \geqslant 4$. Graph theory terminology not presented here can be found in [3].

2. Steiner distance stable graphs

Let G be a connected graph and S a nonempty set of vertices of G. Then the *Steiner distance* $d_G(S)$ of S is the smallest number of edges in a connected subgraph of G that contains the vertices of S. Such a subgraph must necessarily be a tree and is called a *Steiner tree* for S. The problem of finding the Steiner distance of a set of vertices in a (weighted) connected graph G has received considerable attention in the literature (see, for example, [5, 7]).

The concepts of Steiner distance in graphs and distance stable graphs suggest another generalization of distance stable graphs. For the remainder of this section we assume that k,l,s and m are nonnegative integers with $m \geqslant s \geqslant 2$ and k and l not both zero. If S is a set of s vertices in a connected graph G such that $d_G(S)=m$, then S is called an (s,m)-set. A connected graph G is said to be k-vertex l-edge (s,m)-*Steiner distance stable* if, for every (s,m)-set S of G and every set A consisting of at most k vertices of $G-S$ and at most l edges of G, $d_{G-A}(S)=d_G(S)$. Thus k-vertex l-edge $(2,m)$-Steiner distance stable graphs are the $(k,l,\{m\})$-stable graphs. Note that if S is a set of s vertices such that $d_G(S)=s-1$ then $d_{G-A}(S)=d_G(S)$ for any set A of at most k vertices of $G-S$ and at most l edges of G. For this reason we require that $m \geqslant s$.

For any integers k,l,m and s with $m \geqslant s \geqslant 2$ and k and l not both 0 there exists a k-vertex l-edge (s,m)-Steiner distance stable graph. To see this let G be obtained from $m-1$ disjoint copies of K_{k+l+1}, say $H_1,\ldots,H_{m-1}$, by joining every vertex of H_i to every vertex of H_{i+1} for $1 \leqslant i < m-1$ and then adding a vertex v_0 and joining it to every vertex of H_1 and a vertex v_m and joining it to every vertex of H_{m-1}. It is not difficult to see that G is k-vertex l-edge (s,m)-Steiner distance stable.

The next result shows that if distances of (s,m)-sets in a connected graph are preserved after the deletion of certain numbers of vertices and edges, then so are distances preserved for (s,d)-sets where $d>m$.

Theorem 1. *If a connected graph G is k-vertex l-edge (s, m)-Steiner distance stable, then it is k-vertex l-edge $(s, m+1)$-Steiner distance stable.*

Proof. Let $S=\{u_1, u_2, \ldots, u_s\}$ be an $(s, m+1)$-set and A a set consisting of at most k vertices of $G-S$ and at most l edges of G. Let T be a Steiner tree for S in G. Since $m>s-1$, it follows that T contains a vertex that does not belong to S. Let u_i be an end-vertex of T and v a vertex of $T-S$ such that v is the only vertex not in S on the (unique) u_i-v path in T. Define $S'=\{u_1, u_2, \ldots, u_{i-1}, u_{i+1}, u_{i+2}, \ldots, u_s, v\}$. Then $T-u_i$ is a tree of size m that contains S'. So $d_G(S')<m$. If $d_G(S')<m$, then there exists a Steiner tree H of size at most $m-1$ for S'. Since u_i is adjacent to some vertex of S' (either v or the vertex of S that follows u_i on the u_i-v path in T), $d_G(S)\leqslant m$. However, $d_G(S)=m+1$; so $d_G(S')=m$.

Let $A'=(A-\{v\})$. By hypothesis, $d_{G-A'}(S')=m$. Let T' be a Steiner tree for S' in $G-A'$. Note that T' does not contain u_i, otherwise $d_G(S)\leqslant m$. So, since T' has $m+1$ vertices and $m+1>s$, there exists a vertex $u_j \in S-\{u_i\}$ and $v' \in V(T')-S'$ so that u_j is an end-vertex of T' and v' is the only vertex not in S' on the unique u_j-v' path P' in T'.

We consider two cases.

Case 1: Suppose that u_i is adjacent in G to a vertex w of T' different from u_j. Now let $S''=\{u_1, u_2, \ldots, u_{j-1}, u_{j+1}, \ldots, u_s, v'\}$. Since $T'-u_j$ together with u_i and the edge $u_i w$ produces a tree of size m that contains S'', it follows that $d_G(S'')\leqslant m$. Note that the vertex following u_j on the u_j-v' path P' belongs to S'', call it x. If $d_G(S'')<m$, then $d_G(S)\leqslant m$. Hence $d_G(S'')=m$. Since $S''\cap A=\emptyset$, the hypothesis implies that $d_{G-A}(S'')=m$. Let T'' be a Steiner tree for S'' in $G-A$. We have already observed that u_j is adjacent with some vertex x of T''. So T'' together with u_j and the edge $u_j x$ produces a tree of size $m+1$ in $G-A$ that contains S.

Case 2: Suppose that the only vertex of T' to which u_i is adjacent is u_j. Then there is some $u_n \in S$ that is an end-vertex of T' where $u_n \neq u_j$. Otherwise T' is a path with u_j and v as end vertices. However, then $T'-v$ together with u_i and the edge $u_i u_j$ produces a tree of size m that contains S.

Case 2.1: Suppose that there exists an end-vertex u_n of T' such that $u_n \neq u_j$ and such that the vertex w adjacent with u_n is not v. If $w=v'$ or $w \in S$, let $S''=\{u_1, u_2, \ldots, u_{n-1}, u_{n+1}, \ldots, u_s, v'\}$; otherwise let $S''=\{u_1, u_2, \ldots, u_{n-1}, u_{n+1}, \ldots, u_s, w\}$. In either case $T'-u_n$ together with u_i and the edge $u_i u_j$ produces a tree of size m that contains S''. So $d_G(S'')\leqslant m$. Note that if $T_{S''}$ is any Steiner tree for S'' (in either one of the two cases), then u_n is adjacent with some vertex of $T_{S''}$, since u_n is adjacent with a vertex of S''. Therefore $d_G(S'')\geqslant m$, that is, $d_G(S'')=m$. Since $A\cap S''=\emptyset$, it follows from the hypothesis that $d_{G-A}(S'')=d_G(S'')=m$. Let T'' be any Steiner tree for S'' in $G-A$. Since u_n is adjacent to a vertex of S'' a Steiner tree of size $m+1$ for S can now be produced in $G-A$.

Case 2.2: Suppose that every end-vertex different from u_j in T' is adjacent with v. Let u_n be an end-vertex of T' that is adjacent with v. Observe that if $U=\{u_1, u_2, \ldots, u_{n-1}, u_{n+1}, \ldots, u_s, v\}$, then $d_G(U)=m$. Let $A''=(A-\{v\})\cup\{u_n\}$ if $v \in A$ and $A''=A$, otherwise. By hypothesis, $d_{G-A''}(U)=m$. Let T_U be a Steiner tree for U in

$G-A''$. Note that T_U does not contain u_n. So since T_U has $m+1$ vertices and $m+1>s$, there exist vertices $u_r \in S-\{u_n\}$ and $v'' \in V(T_U)-U$, so that u_r is an end-vertex of T_U and v'' is the only vertex not in U on the unique u_r–v'' path P'' in T_U. Observe that u_r is adjacent to a vertex of T_U other than u_n, since u_r is adjacent with v. We now proceed as in Case 1 to show that $d_{G-A}(S)=m+1$, in this case also.

Corollary 1. *If a connected graph is k-vertex l-edge (s, m)-Steiner distance stable, then it is k-vertex l-edge (s, n)-Steiner distance stable for all $n \geqslant m$.*

The next theorem implies another result of this type.

Theorem 2. *If a connected graph G is k-vertex l-edge (s, m)-Steiner distance stable, $m \geqslant s \geqslant 3$, then G is k-vertex l-edge $(s-1, m)$-Steiner distance stable.*

Proof. Let S be a set of $s-1$ vertices in G such that $d_G(S)=m$. Suppose that T_S is a Steiner tree for S. Let $v \in V(T_S)-S$ and define $S'=S \cup \{v\}$. Then $d_G(S')=m$. Let A be any set of at most k vertices of $G-S$ and at most l edges of G. Let $A'=A-\{v\}$. By the hypothesis $d_{G-A'}(S')=d_G(S')=m$. Let $T_{S'}$ be a Steiner tree for S' in $G-A'$ and let $v' \in V(T_{S'})-S'$. Such a vertex exists since $T_{S'}$ has $m+1$ $(>s)$ vertices. Now let $S''=S \cup \{v'\}$ and observe that $S'' \cap A=\emptyset$. Further $d_G(S'')=m$ and $|S''|=s$. So by the hypothesis $d_{G-A}(S'')=d_G(S'')=m$. If $T_{S''}$ is a Steiner tree for S'' in $G-A$, then $T_{S''}$ is also a Steiner tree for S in $G-A$, that is, $d_{G-A}(S)=d_G(S)=m$. $\square$

Corollary 2. *If a connected graph is k-vertex l-edge (s, m)-Steiner distance stable, $m \geqslant s \geqslant 3$, then it is k-vertex l-edge (s', m)-Steiner distance stable for all s' $(2 \leqslant s' \leqslant s)$.*

In Theorem 1 we saw that the condition that a connected graph is k-vertex l-edge (s, m)-Steiner distance stable is sufficient for the graph to be k-vertex l-edge $(s, m+1)$-Steiner distance stable. The next result shows that this condition in *not* necessary.

Theorem 3. *For any integers k, s and m such that $s \geqslant 2$, $2s-2 \geqslant m \geqslant s$ and $k \geqslant 1$, there exists a graph G which is k-vertex 0-edge $(s, m+1)$-Steiner distance stable, but not k-vertex 0-edge (s, m)-Steiner distance stable.*

Proof. Let $r=s+(m-s+3)k+k-1$. Define G to be the kth power C_r^k of the cycle C_r. That is, G is obtained from C_r by joining every vertex u on C_r to every vertex v on C_r with $d_{C_r}(u, v) \leqslant k$. Suppose C_r: $v_0, v_1, \ldots, v_{r-1}, v_0$.

We show first that G is k-vertex 0-edge $(s, m+1)$-Steiner distance stable. Let S be a set of s vertices of G such that $d_G(S)=m+1$. We show that the subgraph $\langle S \rangle_G$ induced by S contains at least $m-s+3$ components. Suppose that $\langle S \rangle_G$ has $m-s+i$ components where $i \geqslant 4$. Then G has at least $(m-s+i)k+s$ vertices. But $(m-s+i)k+s \geqslant s+(m-s+3)k+k$, contrary to the assumption that G has $s+(m-s+3)k+k-1$ vertices. However, since G is a power of a cycle, each vertex in a Steiner tree T_S for S that does not belong to S has degree 2 in T_S and is joined to one vertex from each of two components of $\langle S\rangle_G$. Hence $\langle S\rangle_G$ has at most $m-s+3$ components, otherwise $d_G(S)>m+1$. Thus $\langle S\rangle_G$ has exactly $m-s+3$ components $G_1, G_2, \dots, G_{m-s+3}$. Since G is a power of a cycle, each G_i has a spanning path. Let p_i be the order of G_i for $1\leqslant i\leqslant m-s+3$. Let $v_{i,1}, v_{i,2}, \dots, v_{i,p_i}$ denote the order of the vertices of G_i as they appear on C_r as we proceed in clockwise order about C_r. We may assume that $G_1, G_2, \dots G_{m-s+3}$ are labelled so that v_{i,p_i} precedes $v_{i+1,1}$ on C_r as we proceed in clockwise order about C_r for $1\leqslant i<m-s+3$. Since $G_1, G_2, \dots, G_{m-s+3}$ are components of $\langle S\rangle_G$ it follows that $d_{C_r}(v_{i,p_i}, v_{i+1,1})\geqslant k+1$ for $1\leqslant i\leqslant m-s+2$ and $d_{C_r}(v_{m-s+3,p_{m-s+3}}, v_{1,1})\geqslant k+1$. Let $U_i=N_G(v_{i,p_i})\cap N_G(v_{i+1,1})$ for $1\leqslant i\leqslant m-s+2$ and $U_{m-s+3}=N_G(v_{m-s+3,p_{m-s+3}})\cap N_G(v_{1,1})$. Then $|U_i|\geqslant 1$ for all i since $r=s+(m-s+3)k+k-1$. Also if for some i $(1\leqslant i\leqslant m-s+3)$, $|U_i|=l$, then $d(v_{i,p_i}, v_{i+1,1})=2k-l+1$ (where $v_{m-s+4,1}=v_{1,1}$). So for $j\neq i$, we have $d(v_{j,p_j}, v_{j+1,1})\leqslant k+l$, so that $|U_j|\geqslant k-l+1$. It remains to be seen that if A is a set of at most k vertices that contains all the vertices of some U_i, then $U_j-A\neq\emptyset$ for all $j\neq i$ and $1\leqslant j\leqslant m-s+3$. Hence $d_{G-A}(S)=d_G(S)=m+1$.

We show next that G is not k-vertex 0-edge (s,m)-Steiner distance stable.

Let $S=\{v_0, v_{(k+1)}, v_{2(k+1)}, \dots, v_{(m-s)(k+1)}, v_{(m-s+1)(k+1)}, v_{(m-s+1)(k+1)+1}, \dots, v_{(m-s+1)(k+1)+(2s-m-2)}\}$. Then $|S|=s$ and $d_G(S)=m$. If $A=\{v_1, v_2, \dots, v_k\}$, then $d_{G-A}(S)=m+1>d_G(S)$. So G is not k-vertex 0-edge (s,m)-Steiner distance stable. □

If we let $m=s-1$ in the construction of the proof of Theorem 3, we obtain a graph that is k-vertex 0-edge (s,s)-Steiner distance stable and not k-vertex 0-edge $(s-1,s-1)$-Steiner distance stable.

We now show that the converse of Theorem 2 does not hold.

Theorem 4. *For $s\geqslant 3$ there is a graph which is 1-vertex 0-edge $(s-1,s-1)$-Steiner distance stable but not 1-vertex 0-edge (s,s)-Steiner distance stable.*

Proof. Let $G_1\cong K_{1,s}$ where $V(G_1)=\{x, v_1, v_2, \dots, v_s\}$ and x has degree s in G_1. Let $G_2\cong K_s$ with $V(G_2)=\{w_1, w_2, \dots, w_s\}$. Define G to be the graph obtained from $G_1\cup G_2$ by adding the edges in $\{v_iw_j \mid 1\leqslant i,j\leqslant s \text{ and } i\neq j\}$. Then $S=\{v_1, v_2, \dots, v_s\}$ has Steiner distance s in G but in $G-x$, S has Steiner distance $s+1$. Hence G is not 1-vertex 0-edge (s,s)-Steiner distance stable.

Suppose now that S' is any $(s-1,s-1)$-set. Then $\langle S'\rangle_G$ is not connected. If $|S'\cap V(G_2)|\geqslant 2$, then $|S'\cap\{v_1, v_2, \dots, v_s\}|=0$. Moreover, $x\in S'$. Let $S'=\{x, w_3, \dots, w_s\}$. After deleting any vertex u of G not in S', either v_1 or v_2 still belongs to the resulting graph. Thus in this case $d_{G-u}(S')=d_G(S')=s-1$ for all $(s-1,s-1)$-sets S' and for all $u\in V(G)-S'$.

If $|S'\cap V(G_2)|=1$, say $w_1\in S'$ and $x\in S'$, then necessarily $S'\cap\{v_1, v_2, \dots, v_s\}\subseteq\{v_1\}$. After the deletion of any vertex u from G that does not belong to S', either v_2 or v_3 still belongs to the resulting graph, which implies that $d_{G-u}(S')=d_G(S')=s-1$ in this case also.

If $|S' \cap V(G_2)| = 1$, say $w_1 \in S'$ and $x \notin S'$, then $v_1 \in S'$. After deleting any vertex $u \in V(G) - S'$ from G a graph results that contains either w_2 or w_3. So $d_{G-u}(S') = d_G(S')$ in this case as well.

Finally suppose $S' \subseteq \{v_1, v_2, \ldots, v_n\}$. Then we may assume $v_1 \notin S'$. After deleting any vertex $u \in V(G) - S'$ from G, a graph results that contains either w_1 or x. So in this case $d_{G-u}(S') = d_G(S')$. □

Since the graph G of the proof of Theorem 4 is 1-vertex 0-edge $(s-1, s-1)$-Steiner distance stable, it follows, by Theorem 1, that G is 1-vertex 0-edge $(s-1, s)$-Steiner distance stable. Since G is not 1-vertex 0-edge (s, s)-Steiner distance stable, it follows that the converse of Theorem 2 does not hold in general.

Recall it was shown in [4], for a positive integer k, that a graph is $(k, 0, \{2\})$-stable if and only if it is $(0, k, \{2\})$-stable. So a graph is k-vertex 0-edge (2,2)-Steiner distance stable if and only if it is 0-vertex k-edge (2,2)-Steiner distance stable. The next result shows that the necessity of this condition has an extension to (3,3)-sets.

Theorem 5. *For a positive integer k, a graph G is k-vertex 0-edge (3,3)-Steiner distance stable if it is 0-vertex k-edge (3,3)-Steiner distance stable.*

Proof. Suppose G is not k-vertex 0-edge (3,3)-Steiner distance stable. Let S be a (3,3)-set for which there exists a set A of at most k vertices in $G-S$ such that $d_{G-A}(S) > d_G(S) = 3$. Let $A = \{v_1, v_2, \ldots, v_l\}$ be a minimal such set. Observe that $\langle S \rangle_G$ is not connected, since $d_G(S) = 3$. Let u be a vertex of S that is not adjacent to either of the remaining two vertices of S.

Let $A_i = A - \{v_i\}$ for $1 \leqslant i \leqslant l$. By our choice of A, $d_{G-A_i}(S) = d_G(S) = 3$. So necessarily $uv_i \in E(G)$ for all i. Moreover, every Steiner tree for S in G must contain one of the edges in $B = \{uv_1, uv_2, \ldots, uv_l\}$. Thus $d_{G-B}(S) > d_G(S)$, implying that G is not 0-vertex l-edge (3,3)-Steiner distance stable. □

The converse of Theorem 5 does not hold. Let $H_1 \cong K_s - uv$ $(s \geqslant 3)$ for some pair u, v of vertices and $H_2 \cong K_2$ where $V(H_2) = \{x, y\}$. Let G be obtained from $H_1 \cup H_2$ by adding the edges ux and vy. Then G is 1-vertex 0-edge (s, s)-Steiner distance stable, but G is not 0-vertex 1-edge (s, s)-Steiner distance stable. To see this let $z \in V(H_1) - \{u, v\}$. Then $d_G(\{x, y, z\}) = 3$ but $d_{G-xy}(\{x, y, z\}) = 4$.

Theorem 5 cannot be extended to (s, s)-sets for $s \geqslant 4$. To see this, let $G \cong (K_2 \cup K_{s-2}) + K_1$, i.e., G is obtained by joining a new vertex to every vertex in disjoint copies of K_2 and K_{s-2}. Then G is 0-vertex 1-edge (s, s)-Steiner distance stable but G is not 1-vertex 0-edge (s, s)-Steiner distance stable.

3. Independent Steiner distance stable graphs

In this section we focus our attention on independent sets of vertices of a graph. Our first result shows that in a certain sense the problem of finding Steiner trees for sets of independent vertices is equivalent to the problem of finding the Steiner trees of sets of vertices that are not necessarily independent. Let Π_1 be the problem of finding a Steiner tree for a nonempty set of vertices of a connected graph and Π_2 the problem of finding a Steiner tree for a nonempty independent set of vertices of a connected graph. Let G be a connected graph and S a nonempty set of vertices of G. Suppose $G_1, G_2, \ldots, G_n$ are the components of $\langle S\rangle_G$. Let $R(G;S)$ be the graph with vertex set $(V(G)-S)\cup\{v_1, v_2, \ldots, v_n\}$ (where v_i corresponds to G_i, $1\leqslant i\leqslant n$) and edge set $\{uv \mid uv\in E(G-S)\}\cup\{uv_i \mid u\in V(G-S)$ and u is adjacent in G to some vertex of $G_i\}$. Thus $R(G;S)$ is the contraction of G that results from the partition $(\bigcup_{i=1}^n V(G_i))\cup\{u \mid \in V(G-S)\}$.

Theorem 6. *There is an (efficient) algorithm that solves Π_1* is and only if *there is an (efficient) algorithm that solves Π_2.*

Proof. Clearly if there is an algorithm that solves Π_1 then in particular such an algorithm finds a Steiner tree for an independent set of vertices. So there is an algorithm for solving Π_2.

Suppose now that there is an algorithm for solving Π_2. Let G be a connected graph and S a nonempty set of vertices. Suppose $G_1, G_2, \ldots, G_n$ are the components of $\langle S\rangle_G$. Let v_i correspond to G_i in $R(G;S)$. Observe that $\{v_1, v_2, \ldots, v_n\}$is an independent set of vertices in $R(G;S)$. So by applying the algorithm that solves Π_2 to $R(G;S)$ we obtain a Steiner tree T for $\{v_1, v_2, \ldots, v_n\}$ in $R(G;S)$. Let T' be the tree obtained from T by replacing each v_i with a spanning tree of G_i $(1\leqslant i\leqslant n)$ and every edge of the type uv_i where $u\in V(G-S)$ with an edge of G that joins u to some vertex of G_i. Then T' is a tree that contains S. So $|E(T')|\geqslant d_G(S)$. We show next that T' is a Steiner tree for S.

Let T_S be a Steiner tree for S such that the number of components m of $\langle S\rangle_{T_S}$ is a minimum. Clearly $m\geqslant n$. We show that $m=n$ by showing that $\langle V(G_i)\rangle_{T_S}$ is connected for each i $(1\leqslant i\leqslant n)$. Suppose that $\langle V(G_i)\rangle_{T_S}$ contains at least two components. Let T_1 and T_2 be two such components such that an edge e of G_i (that does not belong to T_S) joins a vertex x of T_1 to a vertex y of T_2. Since x and y belong to S, T_S contains an x–y path P. So P together with the edge e produces a cycle C. This cycle must contain a vertex and hence an edge that does not belong to any G_j $(1\leqslant j\leqslant n)$. Let f be such an edge. Then $T'_S=T_S+xy-f$ is a Steiner tree for S such that $\langle S\rangle_{T'_S}$ has fewer components than $\langle S\rangle_{T_S}$. This contradicts our assumption. Thus $m=n$ and $|E(\langle V(G_i)\rangle_{T_S})|=|V(G_i)|-1$ for $1\leqslant i\leqslant n$. Since $R(T_S;S)$ is a tree of $R(G;S)$ that contains S, it follows that $|E(R(T_S;S))|\geqslant|E(T)|$. So $|E(T')|=|E(T)|+\sum_{i=1}^n(|V(G_i)|-1)\leqslant|E(R(T_S;S))|+\sum_{i=1}^n(|V(G_i)|-1)=|E(T_S)|=d_G(S)$. Thus by an earlier remark $|E(T')|=d_G(S)$. $\square$

Theorem 6 also follows directly from the nearest vertex reduction test described by Beasley [2].

The concepts presented in Section 2 and Theorem 6 suggest the next topic. If G is a connected graph and S an independent set of s vertices of G such that $d_G(S)=m$, then

S is called an $I(s,m)$-set. A connected graph is defined to be *k-vertex l-edge $I(s,m)$-Steiner distance stable* if, for every $I(s,m)$-set S and every set A of at most k vertices of $G-S$ and at most l edges of G, $d_{G-A}(S)=m$. The next result establishes an analogue of Theorem 1 with respect to $I(3,m)$-sets.

Theorem 7. *If G is a k-vertex l-edge $I(3,m)$-Steiner distance stable graph $m\geqslant 4$, then G is a k-vertex l-edge $I(3,m+1)$-Steiner distance stable graph.*

Proof. Let $U=\{u_1,u_2,u_3\}$ be an $I(3,m+1)$-set and let A be a set of at most k vertices of $G-S$ and at most l edges of G. Let T be a Steiner tree for U. Since $m\geqslant 4$ there exists an end-vertex u_i of T such that $d_G(u_i,u_j)\geqslant 3$ for all $u_j\in U-\{u_i\}$. Suppose $u_i=u_1$. Let v be the vertex adjacent with u_1 in T and let $U'=(U-\{u_1\})\cup\{v\}$. Then U' is an $I(3,m)$-set. Let $A'=A-v$. Then $d_{G-A'}(U')=m$. We now consider two cases.

Case 1: Whenever T' is a Steiner tree for U' in $G-A'$, then $N_{T'}(u_2)\cap N_{T'}(u_3)=\emptyset$. Let T' be a Steiner tree for U' in $G-A'$ and v' a vertex adjacent with u_2 in T'. Then $U''=\{u_1,u_3,v'\}$ is an $I(3,m)$-set. Since $v'\notin A$, it follows that there is a Steiner tree of size m for U'' in $G-A$. Such a tree together with u_2 and the edge u_2v' produces a Steiner tree for U of size $m+1$ in $G-A$.

Case 2: There exists a Steiner tree T' for U' in $G-A'$ such that $N_{T'}(u_2)\cap N_{T'}(u_3)\neq\emptyset$. Let $y\in N_{T'}(u_2)\cap N_{T'}(u_3)$. Observe, in this case since $d_G(U)=m+1\geqslant 5$, that $d_G(u_1,u_j)\geqslant m-1$ for $j=2,3$. Note further that $d_G(u_1,u_2)$ and $d_G(u_1,u_3)$ cannot both be $m+1$. So suppose $d_G(u_1,u_2)\leqslant m$.

Case 2.1: Suppose $d_G(u_1,u_2)=m$. Let z be the vertex adjacent with v in T'. Then $U_1=\{u_1,z,u_2\}$ is an $I(3,m)$-set. Note that $z\notin A$ since z belongs to $T'-v$. So there exists a Steiner tree T_1 of size m for U_1 in $G-A$. Since $d_G(u_1,u_2)=m$, the tree T_1 is necessarily a path. Further, $d_{T_1}(u_1,z)=2$. Let v' be a vertex that is adjacent with both u_1 and z in T_1. Then T_1-v together with u_1 and v' and the edges u_1v' and $v'z$ produces a Steiner tree of size $m+1$ in $G-A$ for U.

Case 2.2: Suppose $d(u_1,u_2)=m-1$. Let w be a vertex that is adjacent with u_2 on a shortest u_1–u_2 path. Let $A''=A-\{w\}$. Let $U_2=\{u_1,w,u_3\}$. Then $d_G(U_2)\leqslant m+1$. If $d_G(U_2)=m$, then there exists a Steiner tree T_2 of size m for U_2 in $G-A''$. Let v' be adjacent with u_1 in T_2. Then $v'\notin A$. Note that neither u_3 nor u_2 is adjacent with v'. Also a shortest v'–u_2 path together with the path u_2,y,u_3 produces a connected graph of size at most m that contains $U_3=\{u_3,u_2,v'\}$. So $d_G(U_3)\leqslant m$. However, $d_G(U_3)\nless m$ otherwise $d_G(U)\leqslant m$. Let T_3 be a Steiner tree for U_3 of size m in $G-A$. Then T_3 together with u_1 and the edge u_1v' is a Steiner tree of size $m+1$ for U in $G-A$.

Suppose now that $d_G(U_2)=m+1$. Then wu_3 and wy are not edges of G. Since a shortest u_1–w path together with the path w,u_2,y produces a connected graph of size at most m that contains $U_4=\{u_1,w,y\}$, it follows that $d_G(U_4)\leqslant m$. However, $d_G(U_4)\nless m$, otherwise it follows, since u_3 is adjacent with y, that $d_G(U_2)\leqslant m$. So U_4 is an $I(3,m)$-set. Let T_4 be a Steiner tree of size m in $G-A''$. If w is an end-vertex of T_4, then T_4-w together with u_2 and u_3 and the edges u_2y and u_3y produces a Steiner tree of size $m+1$ for U in $G-A$.

Suppose thus that T_4 is a path that contains w as an internal vertex. Then the u_1-w path of T_4 has length $m-2$. Let v' be the vertex adjacent with u_1 on T_4. Then $U_5=\{v',u_2,u_3\}$ is an $I(3,m)$-set and $v'\notin A$. So there exists a Steiner tree T_5 of size m for U_5 in $G-A$. Thus T_5 together with u_1 and the edge u_1v' produces a Steiner tree for U in $G-A$. $\square$

It remains an open problem to determine if a k-vertex l-edge $I(s,m)$-Steiner distance stable graph $m\geqslant 4$, is a k-vertex l-edge $I(s,m+1)$-Steiner distance stable graph where $s\geqslant 4$.

In closing we wish to remark that it is our belief that the problem of determining whether a graph G is k-vertex l-edge (s,m)-Steiner distance stable (for $m\geqslant s\geqslant 3$ and k and s nonnegative integers not both 0) is NP-hard.

Acknowledgement

We wish to thank the referee for pointing out that Theorem 6 also follows directly from the nearest vertex reduction test described by Beasley in [2].

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